A COMPUTATIONAL APPROACH TO UNBIASED DISTRICTING

Joint work with Clemens Puppe

University of Karlsruhe

Attila Tasnádi, Corvinus University of Budapest

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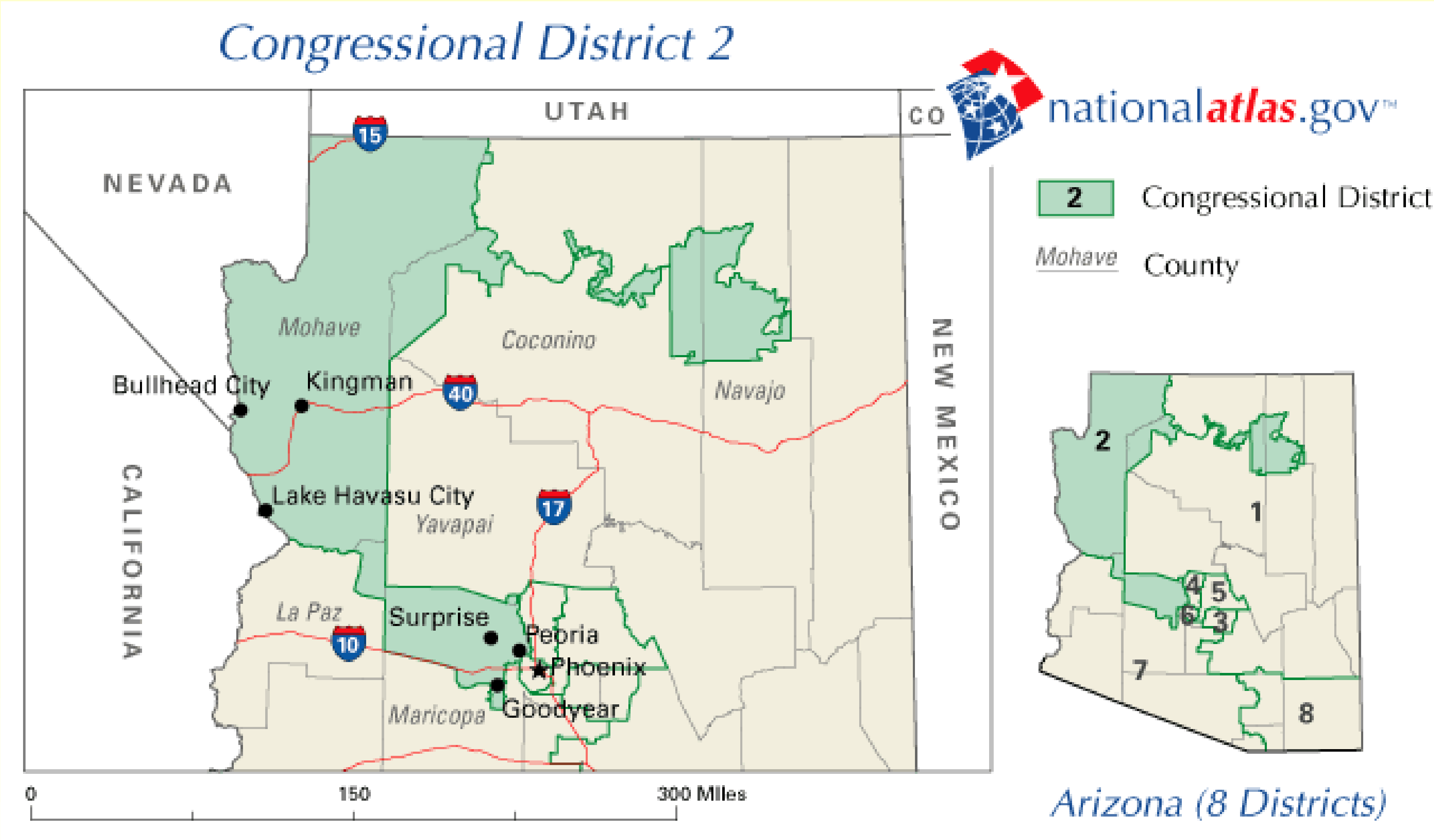
- HU: No redistricting since 1990.
- SA: Done by the Munincipal Demarcation Board.
- US: In most of the states mainly done by the state legislature subject to approval by the state governor.

Arizona



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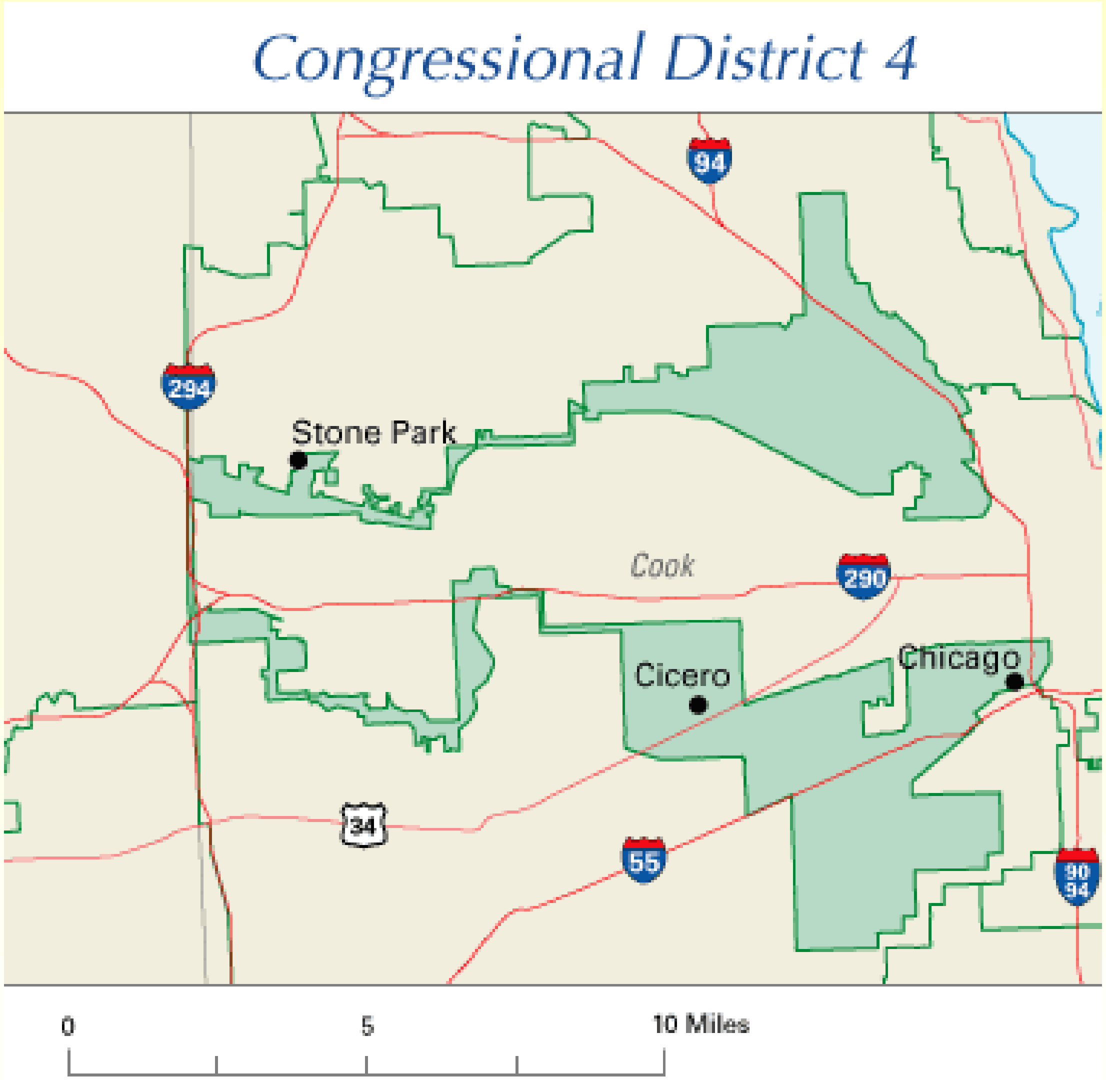

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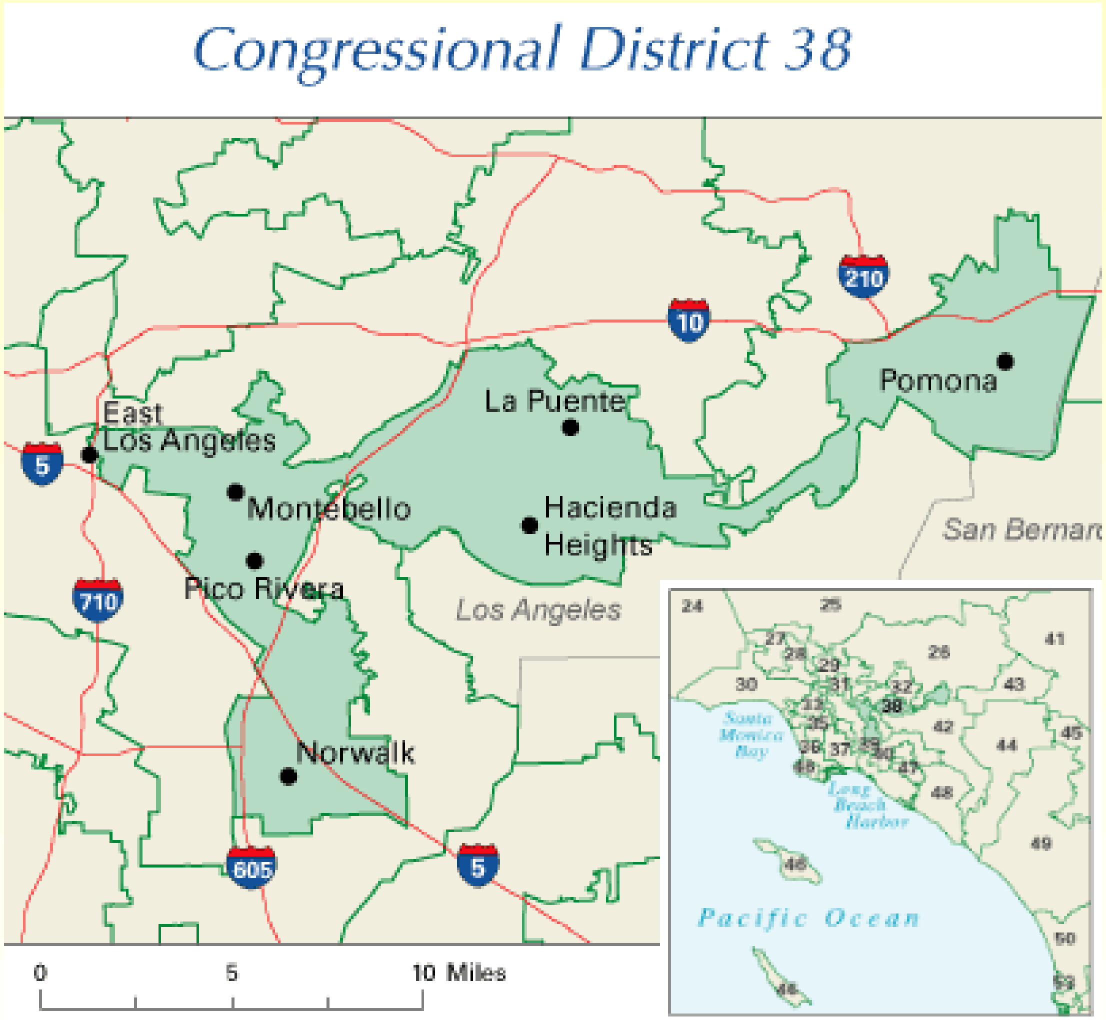

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- Economics: Besley and Preston (2007), Coate and Knight (2007), Gul and Pesendorfer (2007) and Friedman and Holden (2008).

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- Full participation rate.

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Remark: If $\mathcal{S}$ consists of all $2k + 1$ sized subsets of N, we speak of *districting without geographical constraints*.

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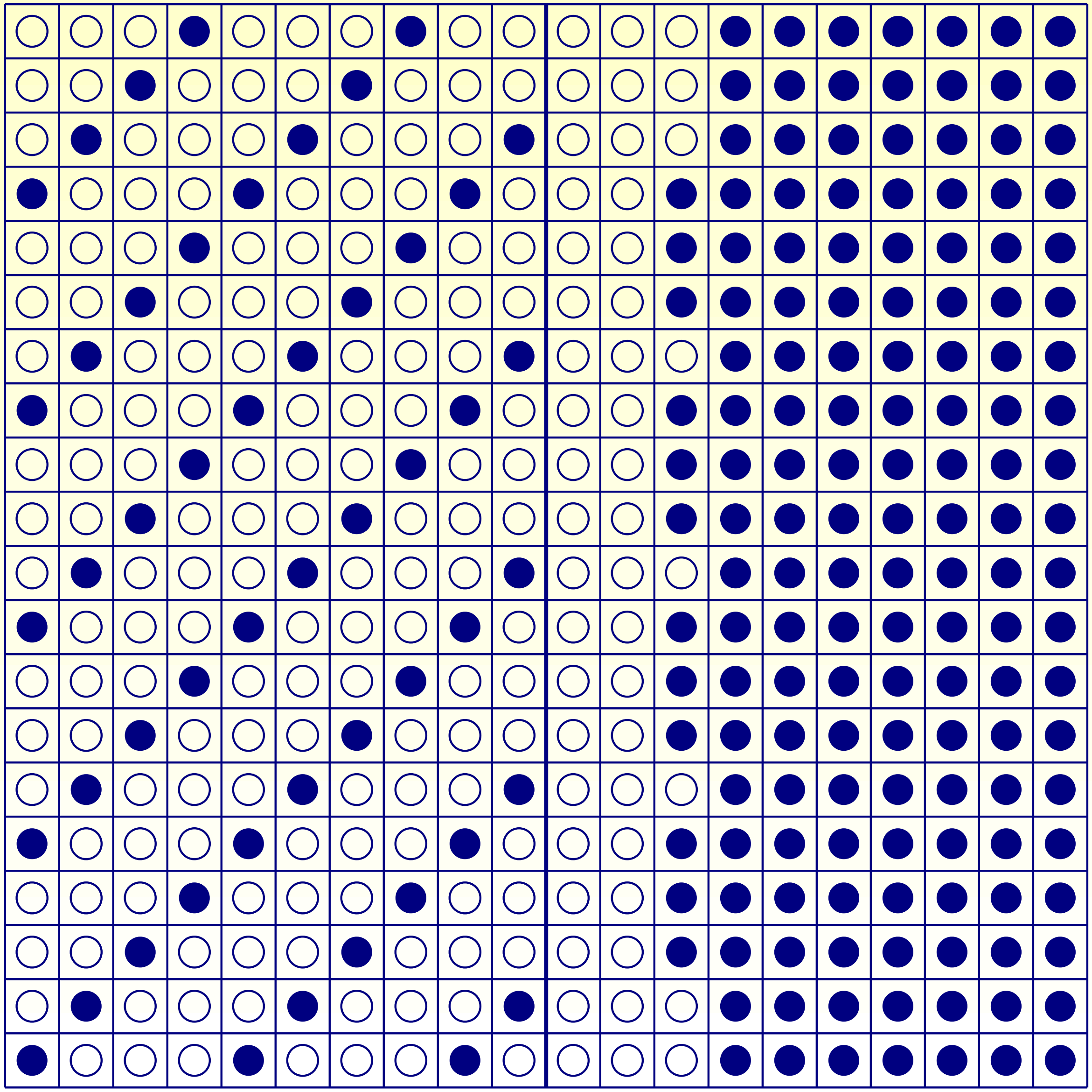
Non-existence of an unbiased districting ($k = 2$)



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5. Proving that f is a polynomial transformation.

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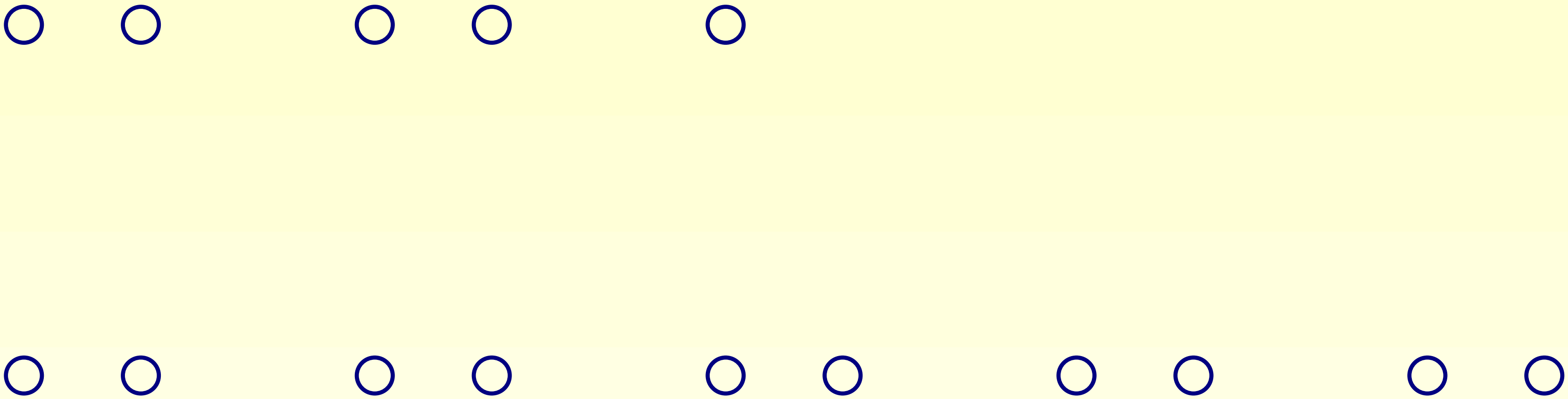
Proof: Unbiasedness of f can be verified in polynomial time (i.e, UNBIASED DISTRICTING $\in$ NP).

Motivating example ($k = 2$, $d = 8$ and $n = 40$)



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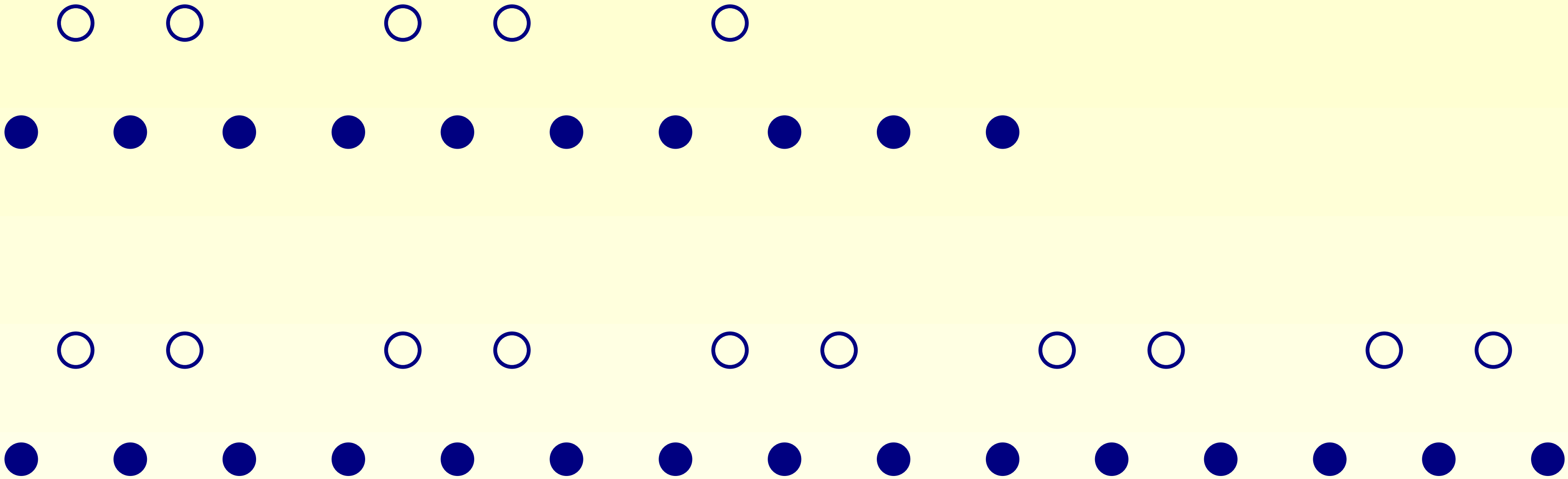

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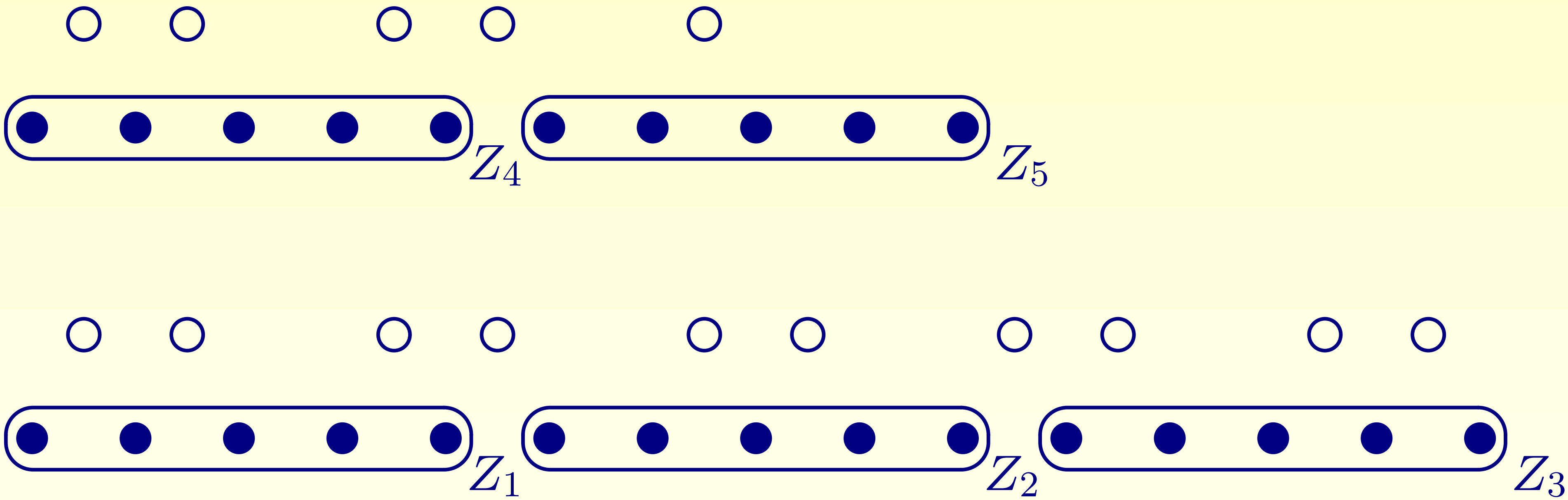

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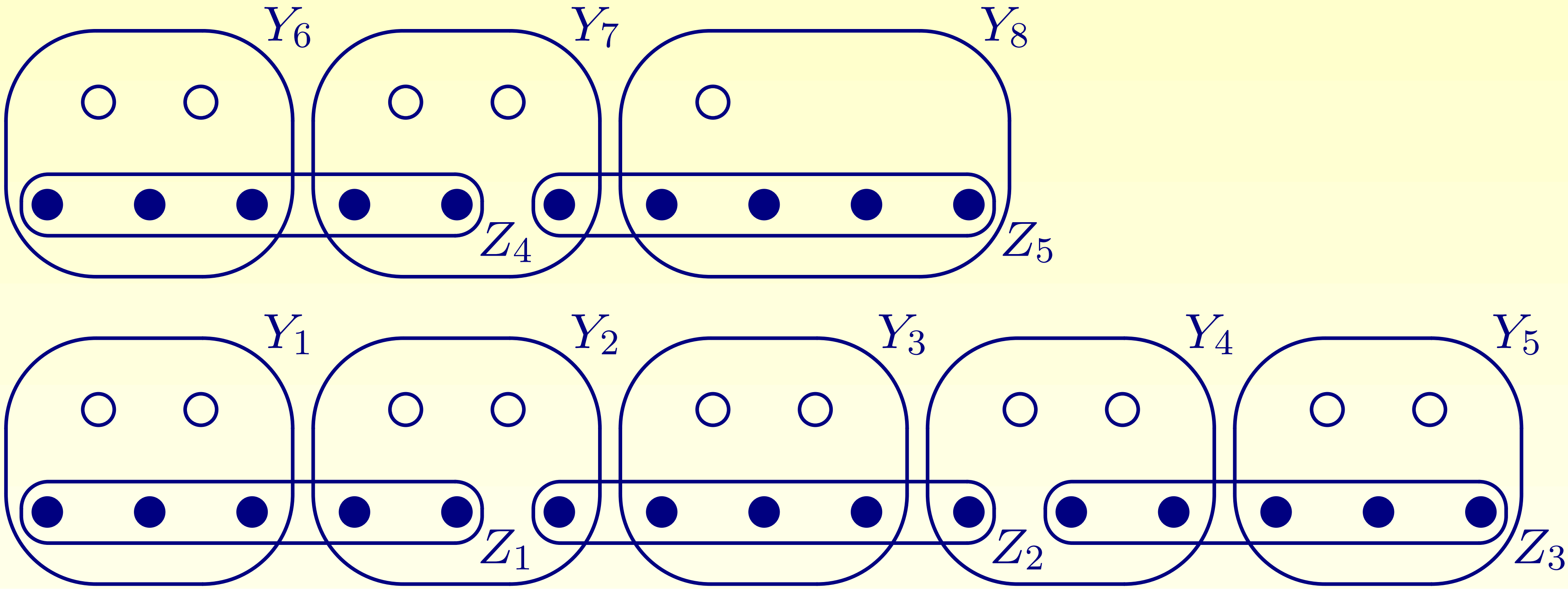

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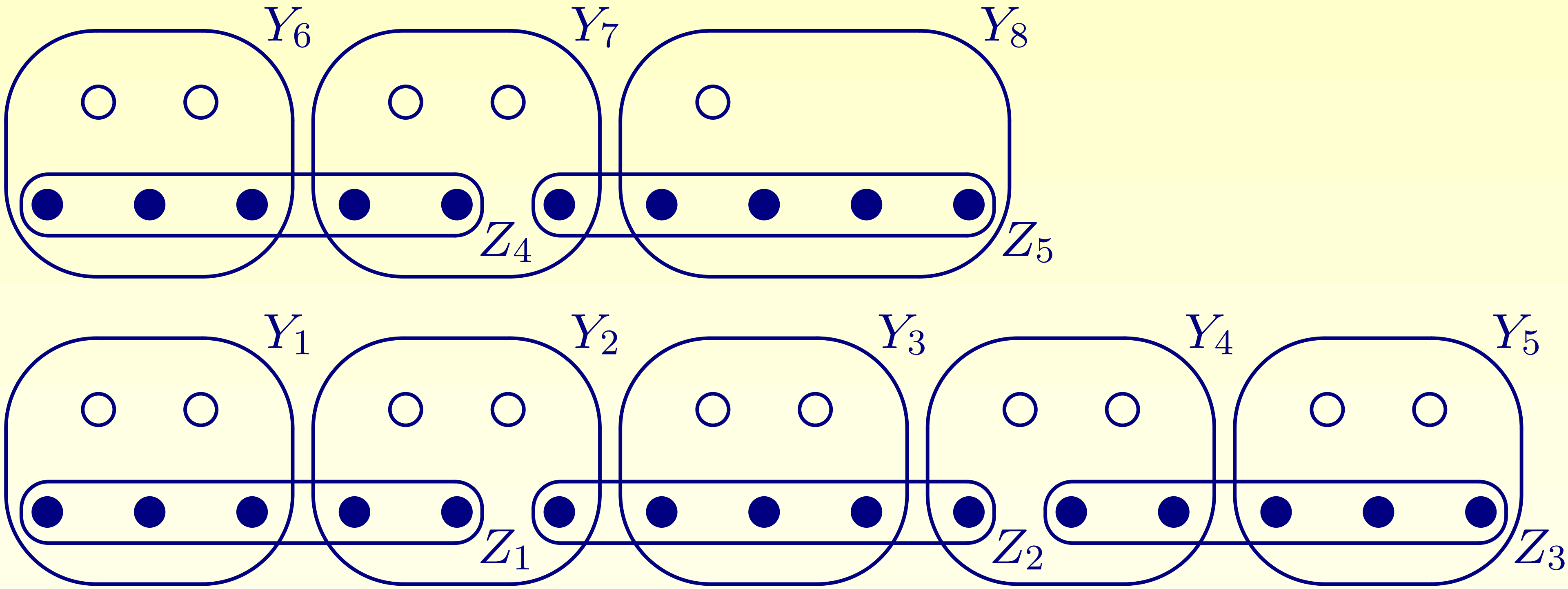

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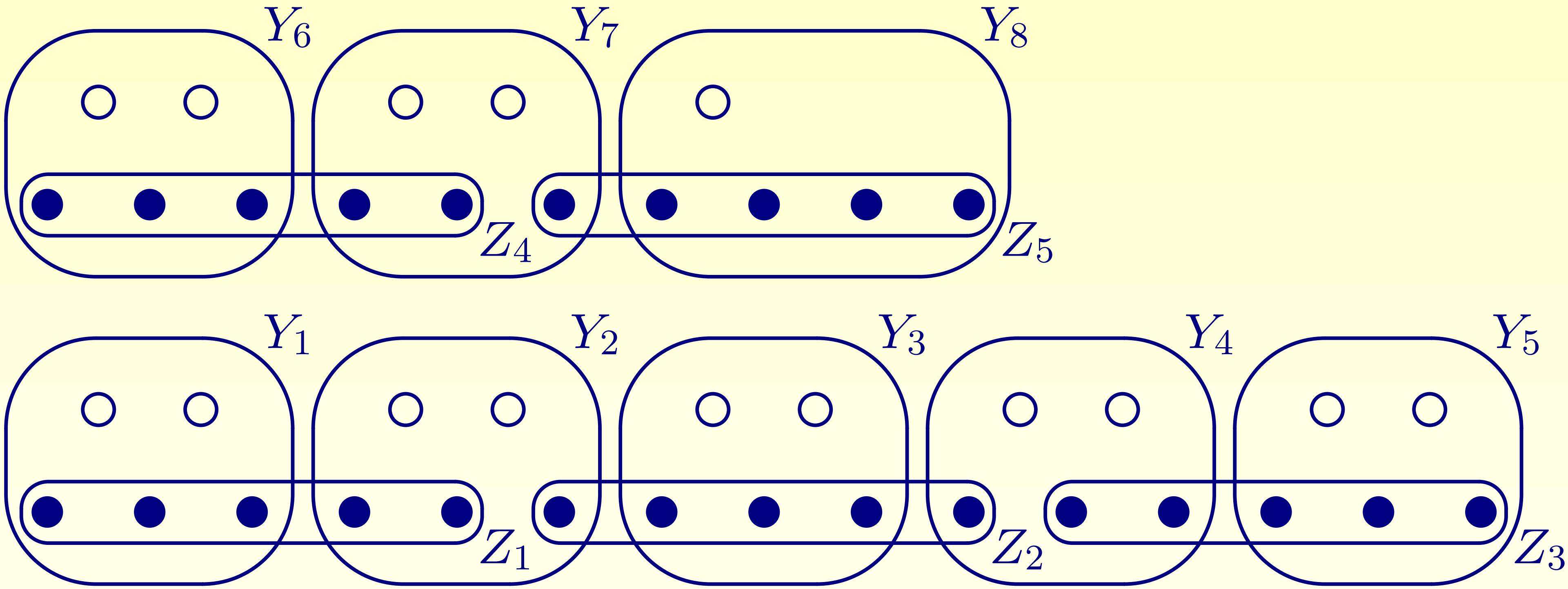

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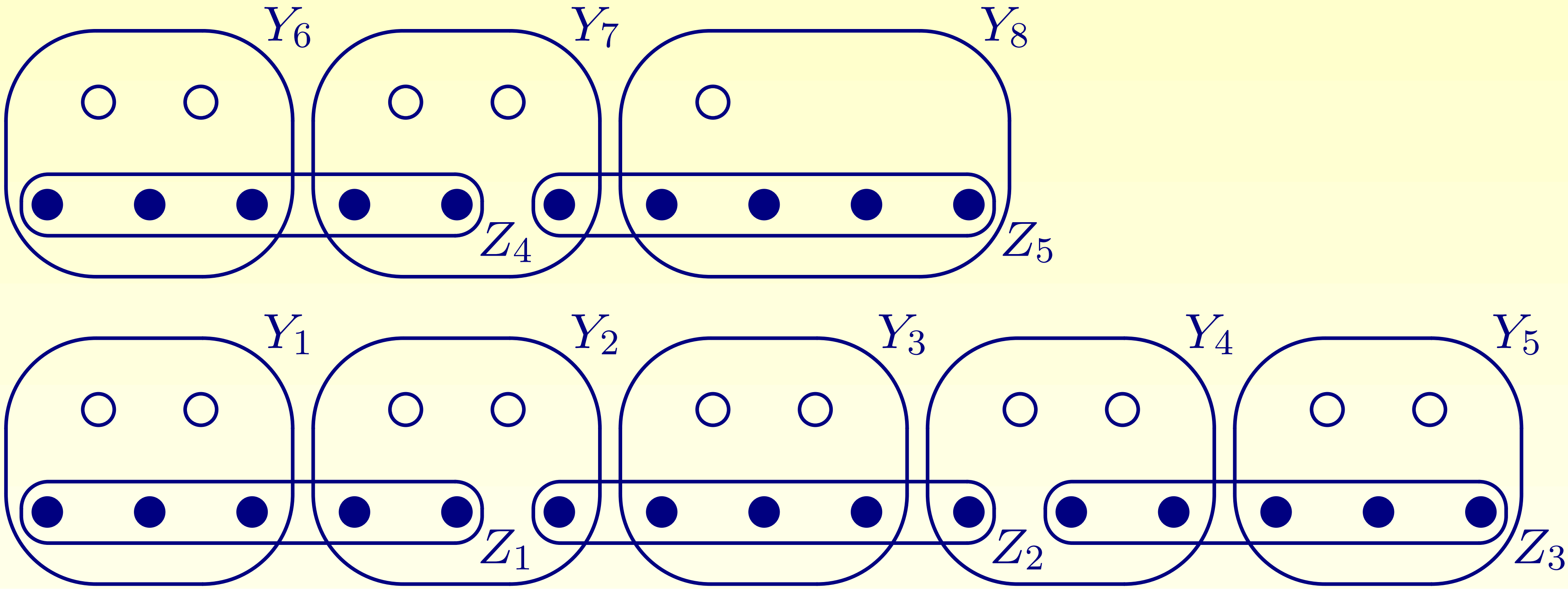

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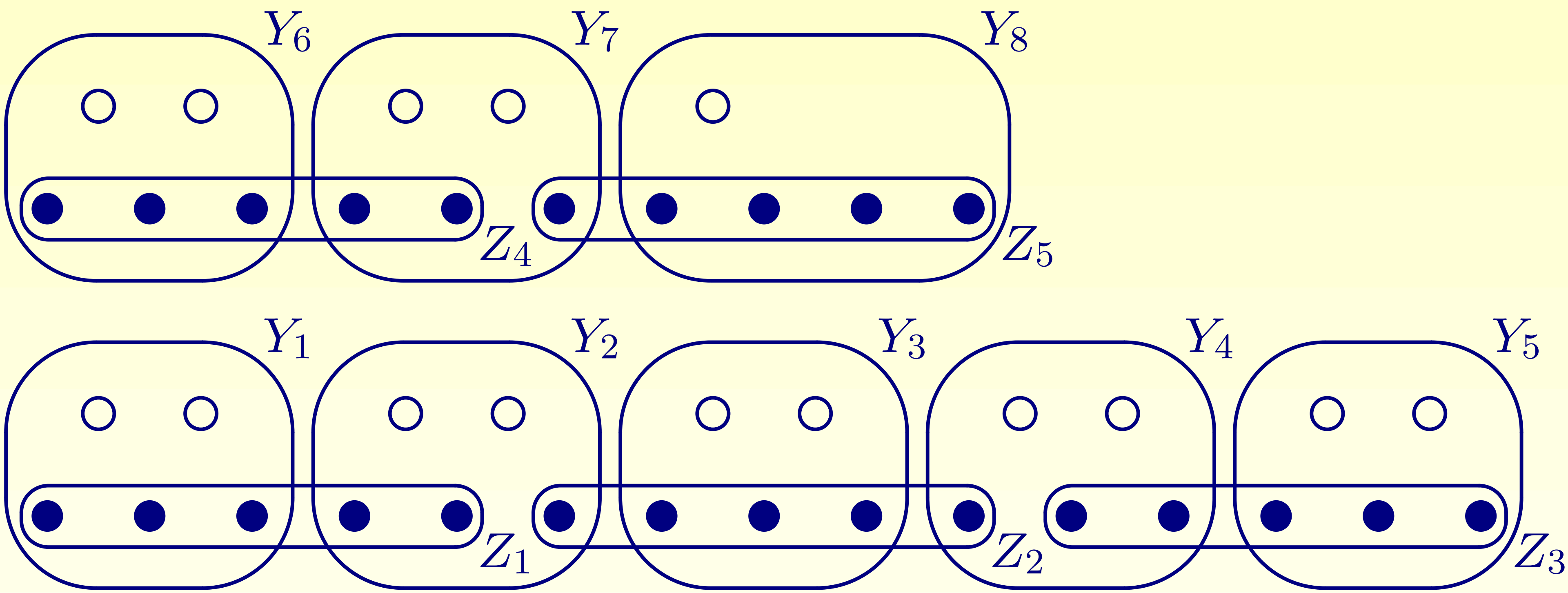

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The districting is unbiased iff $\mathcal{C}$ has an EXACT COVER.

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